

# LOCAL HARDY-LITTLEWOOD MAXIMAL OPERATOR IN VARIABLE LEBESGUE SPACES

AMIRAN GOGATISHVILI, ANA DANELIA, TENGIZ KOPALIANI

**ABSTRACT.** We investigate the class  $\mathcal{B}^{loc}(\mathbb{R}^n)$  of exponents  $p(\cdot)$  for which the local Hardy-Littlewood maximal operator is bounded in variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$ . Littlewood-Paley square function characterization of  $L^{p(\cdot)}(\mathbb{R}^n)$  spaces with the above class of exponent are also obtained.

## 1. INTRODUCTION

The variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$  and the corresponding variable exponent Sobolev spaces  $W^{k,p(\cdot)}$  are of interest for their applications to the problems in fluid dynamics [24, 25], partial differential equations with non-standard growth condition and calculus of variations [1, 2, 10, 11], image processing [3, 13, 21].

The boundedness of Hardy-Littlewood maximal operator is very important tool to get boundedness of more complicated operators such as singular integral operators, commutators of singular integrals, Riesz potential and many another operators. Conditions for the boundedness of the Hardy-Littlewood maximal operator on variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$  have been studied in [7, 8, 5, 23, 16, 14, 20]. For an overview we refer to the monograph [9].

Let  $p : \mathbb{R}^n \rightarrow [1, \infty)$  be a measurable function. Denote by  $L^{p(\cdot)}(\mathbb{R}^n)$  the space of all measurable functions  $f$  on  $\mathbb{R}^n$  such that for some  $\lambda > 0$

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty,$$

with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Given a locally integrable function  $f$  on  $\mathbb{R}^n$ , the Hardy-Littlewood maximal operator  $M$  is defined by the equality

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes  $Q$  containing  $x$ . Throughout the paper, all cubes are assumed to have their sides parallel to the coordinate axes.

---

2000 *Mathematics Subject Classification.* 46E30, 42B25, 42B20.

*Key words and phrases.* variable exponent Lebesgue space, local Hardy-Littlewood maximal function, local Muckenhoupt classes, Littlewood-Paley theory, square function.

The research was supported by Shota Rustaveli National Science Foundation grant no.13/06 (Geometry of function spaces, interpolation and embedding theorems). The research of the first author was partially supported by the grant 201/08/0383 and 13-14743S of the Grant Agency of the Czech Republic and RVO: 67985840.

Let  $f$  be locally integrable function  $f$  on  $\mathbb{R}^n$ . We consider the local variant of the Hardy-Littlewood maximal operator given by

$$M^{loc}f(x) = \sup_{Q \ni x, |Q| \leq 1} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Denote by  $\mathcal{B}(\mathbb{R}^n)$  ( $\mathcal{B}^{loc}(\mathbb{R}^n)$ ) the class of all measurable functions  $p : \mathbb{R}^n \rightarrow [1, \infty)$  for which operator  $M$  (operator  $M^{loc}$ ) is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . Given any measurable function  $p(\cdot)$ , let  $p_- = \inf_{x \in \mathbb{R}^n} p(x)$  and  $p_+ = \sup_{x \in \mathbb{R}^n} p(x)$ . Below we assume that  $1 < p_- \leq p_+ < \infty$ .

It has been proved by Diening [7] that if  $p(\cdot)$  satisfies the following uniform continuity condition

$$(1.1) \quad |p(x) - p(y)| \leq \frac{c}{\log(1/|x - y|)}, \quad |x - y| < 1/2,$$

and if  $p(\cdot)$  is a constant outside some large ball, then  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . After that the second condition on  $p(\cdot)$  has been improved independently by Cruz-Uribe, Fiorenza, and Neugebauer [5] and Nekvinda [23]. It is shown in [5] that if  $p(\cdot)$  satisfies (1.1) and

$$(1.2) \quad |p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)}$$

for some  $p_\infty > 1$ , then  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . In [23], the boundedness of  $M$  is deduced from (1.1) and the integral condition more general than (1.2) condition: there exist constants  $c, p_\infty$ , such that  $0 < c < 1, p_\infty > 1$ , and

$$\int_{\mathbb{R}^n} c^{\frac{1}{|p(x) - p_\infty|}} dx < \infty.$$

The condition (1.1) is named the local log-Hölder continuity condition and the condition (1.2) the log-Hölder decay condition (at infinity). The conditions (1.1) and (1.2) together are named global log-Hölder continuity condition. these conditions are connected to the geometry of the space  $L^{p(\cdot)}(\mathbb{R}^n)$ .

By  $\mathcal{X}^n$  we denote the set of all open cubes in  $\mathbb{R}^n$  and by  $\mathcal{Y}^n$  ( $\mathcal{Y}_{loc}^n$ ) we denote the set of all families  $\mathcal{Q} = \{Q_i\}$  of disjoint, open cubes in  $\mathbb{R}^n$  (with measure less than 1) such that  $\bigcup Q_i = \mathbb{R}^n$ .

Everywhere below by  $l_{\mathcal{Q}}$  we denote a Banach sequential space (BSS). Let  $\{e_Q\}$  be standard unit vectors in  $l_{\mathcal{Q}}$ .

**Definition 1.1.** Let  $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \mathcal{Y}^n}$  ( $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \mathcal{Y}_{loc}^n}$ ) be a family of BSSs. A space  $L^{p(\cdot)}(\mathbb{R}^n)$  is said to satisfy a uniformly upper (lower)  $l$ -estimate ( $l_{loc}$ -estimate) if there exists a constant  $C > 0$  such that for every  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  and  $\mathcal{Q} \in \mathcal{Y}^n$  ( $\mathcal{Q} \in \mathcal{Y}_{loc}^n$ ) we have

$$\|f\|_{p(\cdot)} \leq C \left\| \sum_{Q_i \in \mathcal{Q}} \|f \chi_{Q_i}\|_{p(\cdot)} \cdot e_{Q_i} \right\|_{l_{\mathcal{Q}}} \left( \left\| \sum_{Q_i \in \mathcal{Q}} \|f \chi_{Q_i}\|_{p(\cdot)} \cdot e_{Q_i} \right\|_{l_{\mathcal{Q}}} \leq C \|f\|_{p(\cdot)} \right).$$

Definition 1.1 was introduced by Kopaliani in [15]. The idea of definition 1.1 is simply to generalize the following property of the Lebesgue-norm:

$$\|f\|_{L^p}^p = \sum_i \|f \chi_{\Omega_i}\|_{L^p}^p$$

for a partition of  $\mathbb{R}^n$  into measurable sets  $\Omega_i$ .

Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . For any  $\mathcal{Q} \in \mathcal{Y}^n$  we define the space  $l^{\mathcal{Q}, p(\cdot)}$  by

$$l^{\mathcal{Q}, p(\cdot)} := \left\{ \bar{t} = \{t_Q\}_{Q \in \mathcal{Q}} : \sum_{Q \in \mathcal{Q}} |t_Q|^{p_Q} < \infty \right\},$$

equipped with the Luxemburg's norm, where the numbers  $p_Q$  are defined as  $\frac{1}{p_Q} = \frac{1}{|\mathcal{Q}|} \int_Q \frac{1}{p(x)} dx$ . Analogously we define the space  $l^{\mathcal{Q}, p'(\cdot)}$  where  $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$ ,  $t \in \mathbb{R}^n$ .

Note that if  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  then for simple functions we have uniformly lower and upper  $l = \{l^{\mathcal{Q}, p(\cdot)}\}_{\mathcal{Q} \in \mathcal{Y}^n}$  estimates.

**Theorem 1.2.** *Let  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  then uniformly*

$$(1.3) \quad \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right\|_{p(\cdot)} \asymp \left\| \sum_{Q \in \mathcal{Q}} t_Q \|\chi_Q\|_{p(\cdot)} e_Q \right\|_{l^{\mathcal{Q}, p(\cdot)}}$$

and

$$(1.4) \quad \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right\|_{p'(\cdot)} \asymp \left\| \sum_{Q \in \mathcal{Q}} t_Q \|\chi_Q\|_{p'(\cdot)} e_Q \right\|_{l^{\mathcal{Q}, p'(\cdot)}}.$$

Above theorem is another version of necessary part of Diening's Theorem 4.2 in [8] (proof may be found in [17]). Note that conditions (1.3) and (1.4) in general do not imply  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . The proof (see in [19]) relies on the example constructed by Lerner in [20]. We give the proof of this fact also here.

Let  $E = \cup_{k \geq 1} (e^{k^3}, e^{k^3 e^{1/k^2}})$  and

$$(1.5) \quad p_0(x) = \int_{|x|}^{\infty} \frac{1}{t \log t} \chi_E(t) dt.$$

There exist  $\alpha > 1$  and  $\beta_0$  ( $1/\alpha < \beta_0 < 1$ ) such that  $p_0(\cdot) + \alpha \in \mathcal{B}(\mathbb{R})$  and  $\beta_0(p_0(\cdot) + \alpha) \notin \mathcal{B}(\mathbb{R})$  (see [20, Theorem 1.7]). Note that for a space  $L^{p(\cdot)}(\mathbb{R})$ , with  $p(\cdot) = p_0(\cdot) + \alpha$  there exists a family  $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \mathcal{Y}^n}$  of BSSs for which  $L^{p(\cdot)}(\mathbb{R})$  satisfies uniformly lower and upper  $l$ -estimate (see [19, Proposition 3.2]). From (1.3) we have  $l_{\mathcal{Q}} \cong l^{\mathcal{Q}, p(\cdot)}$  and consequently we have

$$(1.6) \quad \|f\|_{p(\cdot)} \asymp \left\| \sum_{Q \in \mathcal{Q}} \|f \chi_Q\|_{p(\cdot)} e_Q \right\|_{l^{\mathcal{Q}, p(\cdot)}}.$$

Note that for all  $1 > \beta > \frac{1}{p_-}$

$$(1.7) \quad \|f^{\frac{1}{\beta}}\|_{\beta p(\cdot)}^{\beta} = \|f\|_{p(\cdot)}$$

and

$$(1.8) \quad \|\{t_Q\}\|_{l^{\mathcal{Q}, p(\cdot)}} = \left\| \left\| |t_Q|^{\frac{1}{\beta}} \right\| \right\|_{l^{\mathcal{Q}, p(\cdot)}}^{\beta}.$$

From (1.6), (1.7) and (1.8) we have

$$\|g\|_{\beta p(\cdot)} \asymp \left\| \sum_{Q \in \mathcal{Q}} \|g \chi_Q\|_{p(\cdot)} e_Q \right\|_{l^{\mathcal{Q}, \beta p(\cdot)}}.$$

for  $g \in L^{\beta p(\cdot)}(\mathbb{R})$  and the space  $L^{\beta p(\cdot)}(\mathbb{R})$  satisfies uniformly lower and upper  $l^{\beta}$ -estimates, where  $l_Q^{\beta} = l^{\mathcal{Q}, \beta p(\cdot)}$ .

Note that  $\frac{1}{(\beta p(\cdot))_Q} + \frac{1}{((\beta p(\cdot))'_Q)} = 1$  and  $(l^{\mathcal{Q}, \beta p(\cdot)})' = l^{\mathcal{Q}, (\beta p(\cdot))'}$ . Thus the space  $(L^{\beta p(\cdot)}(\mathbb{R}))'$  satisfies uniformly lower and upper  $(l^\beta)'$ -estimates, where  $(l^\beta)'_Q = l^{\mathcal{Q}, (\beta p(\cdot))'}$  and (1.3) and (1.4) are valid for any  $\beta p(\cdot)$ ,  $(\beta p(\cdot))'$ , where  $1 > \beta > \frac{1}{p_-}$ . Consequently for exponent  $\beta_0 p(\cdot)$  (1.3) and (1.4) are valid but,  $\beta_0 p(\cdot) \notin \mathcal{B}(\mathbb{R})$ .

**Remark 1.3.** Let  $p(\cdot)$  be global log-Hölder continuous function. Then there exists family  $l = \{l_Q\}_{Q \in \mathcal{Q}}$  of BSSs for which  $L^{p(\cdot)}(\mathbb{R}^n)$  satisfies uniformly lower and upper  $l$ -estimates (see [19, Proposition 3.4]). As we already mentioned it was show in [5] that  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  and by Theorem 1.2 (1.3) holds and therefore we have  $l_Q \cong l^{\mathcal{Q}, p(\cdot)}$  and consequently

$$(1.9) \quad \|f\|_{p(\cdot)} \asymp \left\| \sum_{Q \in \mathcal{Q}} \|f \chi_Q\|_{p(\cdot) e_Q} \right\|_{l^{\mathcal{Q}, p(\cdot)}}.$$

**Remark 1.4.** Let  $\mathcal{Q} = \{Q_i\}$  be a partition of  $\mathbb{R}^n$  into equal sizes cubes, ordered so that  $i > j$  if  $\text{dist}(0, Q_i) > \text{dist}(0, Q_j)$ . Let  $p(\cdot)$  be global log-Hölder continuous. Then

$$(1.10) \quad \|f\|_{p(\cdot)} \approx \left( \sum_i \|f \chi_{Q_i}\|_{p(\cdot)}^{p_\infty} \right)^{1/p_\infty}.$$

This was shown in [12, Theorem 2.4]. This statement also follows from Remark 1.3. Indeed, if we have a partition  $\mathcal{Q} = \{Q_i\}$  with equal sizes cubes and it is ordered as above by using [22, Theorem 4.3] we can show that  $l^{p_\infty} \cong l^{\mathcal{Q}, p(\cdot)}$  and consequently from (1.9) we get (1.10).

By  $\mathcal{AC}$  we denote the set of exponents  $p : \mathbb{R} \rightarrow [1, +\infty)$  of the form  $p(x) = p + \int_{-\infty}^x l(u) du$ , where  $\int_{-\infty}^{+\infty} |l(u)| du < +\infty$ .

Note that example of exponent constructed by Lerner and mentioned above belongs to class  $\mathcal{AC}$ . In general we have the following

**Proposition 1.5.** [19, Proposition 3.2] *Let  $p(\cdot) \in \mathcal{AC}$ . Then exists family  $l = \{l_Q\}_{Q \in \mathcal{Q}}$  of BSSs for which  $L^{p(\cdot)}(\mathbb{R}^n)$  satisfies uniformly lower and upper  $l$ -estimate.*

In many applications it is enough to study only boundadnes of local Hardy-Littlewood maximal operator rather the Hardy-Littlewood maximal operator. For example in the Littlewood-Paley theory we need local Hardy-Littlewood maximal operator. In the weighted Lebesgue spaces behavior of local Hardy-Littlewood maximal operator was studied by Rychkov in [26].

In this paper we investigate the class  $\mathcal{B}^{loc}(\mathbb{R}^n)$  of exponents  $p(\cdot)$  for which the local Hardy-Littlewood maximal operator is bounded in variable exponent Lebesgue space  $L^{p(\cdot)}(\mathbb{R}^n)$ . Using the obtained results we give Littlewood-Paley square-function characterization of the variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^n)$  with the above class of exponent.

The paper is organized as follows. In Section 2 we give main results. In section 3 we give application in the Littlewood-Paley theory and in last section we give outlines of the proof of the Theorem 2.2 which is local version of the Dieninges theorem from [8].

## 2. MAIN RESULTS

For any family of pairwise disjoint cubes  $\mathcal{Q}$  and  $f \in L^1_{loc}$  we define the averaging operator

$$T_{\mathcal{Q}}f = \sum_{Q \in \mathcal{Q}} \chi_Q M_Q f$$

where  $M_Q f = |Q|^{-1} \int_Q f(x) dx$ .

We say that exponent  $p(\cdot)$  is of class  $\mathcal{A}$  ( class  $\mathcal{A}^{loc}$ ) if and only if there exists  $C > 0$  such that for all  $\mathcal{Q} \in \mathcal{Y}^n$  ( $\mathcal{Q} \in \mathcal{Y}^n_{loc}$ ) and all  $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\|T_{\mathcal{Q}}\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)},$$

i.e. the averaging operators  $T_{\mathcal{Q}}$  are uniformly continuous on  $L^{p(\cdot)}(\mathbb{R}^n)$ .

A necessary and sufficient condition on  $p(\cdot)$  for which operator  $M$  is bounded in  $L^{p(\cdot)}(\mathbb{R}^n)$  is given by Diening in [8]. It states that  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  if the averaging operators  $T_{\mathcal{Q}}$  are uniformly continuous on  $L^{p(\cdot)}(\mathbb{R}^n)$  with respect to all families  $\mathcal{Q}$  of disjoint cubes. This concept provides the following characterization of when the maximal operator is bounded.

**Theorem 2.1.** ([8, Theorem 8.1]). *Let  $1 < p_- \leq p_+ < \infty$ . The following are equivalent:*

- 1)  $p(\cdot)$  is of class  $\mathcal{A}$ ;
- 2)  $M$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ ;
- 3)  $(M(|f|^q))^{1/q}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$  for some  $q > 1$ , ("left-openness");
- 4)  $M$  is bounded on  $L^{p(\cdot)/q}(\mathbb{R}^n)$  for some  $q > 1$ , ("left-openness");
- 5)  $M$  is bounded on  $L^{p'(\cdot)}(\mathbb{R}^n)$ .

The following theorem is analog of Dienings theorem for local maximal function  $M^{loc}$ . The proof is principally the same as of Theorem 2.1 with minor modification of some facts. The convenience for reader in appendix we sketch of proof.

**Theorem 2.2.** *Let  $1 < p_- \leq p_+ < \infty$ . The following are equivalent:*

- 1)  $p(\cdot)$  is of class  $\mathcal{A}^{loc}$ ;
- 2)  $M^{loc}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ ;
- 3)  $(M^{loc}(|f|^q))^{1/q}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$  for some  $q > 1$ , ("left-openness");
- 4)  $M^{loc}$  is bounded on  $L^{p(\cdot)/q}(\mathbb{R}^n)$  for some  $q > 1$ , ("left-openness");
- 5)  $M^{loc}$  is bounded on  $L^{p'(\cdot)}(\mathbb{R}^n)$ .

We say that  $dx$  satisfies the condition  $A_{p(\cdot)}$  (condition  $A^{loc}_{p(\cdot)}$ ) if there exists  $C > 0$  such that for any cube  $Q$  (for any cube  $Q$  with  $|Q| \leq 1$ )

$$\frac{1}{|Q|} \|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} \leq C.$$

Using Theorem 2.1-2.2 we obtain some subclass of  $\mathcal{B}(\mathbb{R}^n)$  and  $\mathcal{B}^{loc}(\mathbb{R}^n)$ .

**Theorem 2.3.** *Let  $1 < p_- \leq p_+ < \infty$  and there exists family  $l = \{l_Q\}_{Q \in \mathcal{Y}^n}$  (family  $l = \{l_Q\}_{Q \in \mathcal{Y}^n_{loc}}$ ) of BSSs for which  $L^{p(\cdot)}(\mathbb{R}^n)$  satisfies uniformly lower and upper  $l$ -estimate ( $l_{loc}$ -estimate). Then operator  $M$  (operator  $M^{loc}$ ) is bounded in  $L^{p(\cdot)}(\mathbb{R}^n)$  if and only if  $dx \in A_{p(\cdot)}$  ( $dx \in A^{loc}_{p(\cdot)}$ ).*

Proof. The proof for operator  $M^{loc}$  is the same as for operator  $M$ . Let  $dx \in A_{p(\cdot)}^{loc}$ . Using Hölders inequality we get

$$\frac{1}{|Q|} \int_Q |f(x)| dx \leq C \frac{\|f\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}.$$

For  $Q \in \mathcal{Y}_{loc}^n$  and  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  we have

$$\left\| \sum_{Q \in \mathcal{Q}} \chi_Q \frac{1}{|Q|} \int_Q f(x) dx \right\|_{p(\cdot)} \leq \left\| \sum_{Q \in \mathcal{Q}} \|f\chi_Q\|_{p(\cdot)} e_Q \right\|_{l_{\mathcal{Q}}} \leq C \|f\|_{p(\cdot)}.$$

The necessary part of theorem is obvious.

**Theorem 2.4.**  $\mathcal{B}(\mathbb{R}^n) \neq \mathcal{B}^{loc}(\mathbb{R}^n)$

Proof. Let us consider the exponent  $p(\cdot) = \beta_0(p_0(\cdot) + \alpha)$  where  $p_0(\cdot)$  is defined by (1.5). Let fix  $\alpha > 1$  and  $\beta_0$  ( $1/\alpha < \beta_0 < 1$ ) such that the exponent  $p(\cdot)$  does not belong to the class  $\mathcal{B}(\mathbb{R})$ . Since  $p(\cdot) \in \mathcal{AC}$  we can conclude that for  $L^{p(\cdot)}(\mathbb{R})$  there exists family  $l = \{l_Q\}_{Q \in \mathcal{Y}^n}$  of BSSs for which  $L^{p(\cdot)}(\mathbb{R})$  satisfies uniformly lower and upper  $l$ -estimate. For  $p(\cdot)$  the condition (1.1) is fulfilled, so it is easy to show that for this exponent  $p(\cdot)$  condition  $A_{p(\cdot)}^{loc}$  is satisfied. Therefore by Theorem 2.2  $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R})$ .

In the class  $\mathcal{B}^{loc}(\mathbb{R})$  there exist exponents that have arbitrary slow decreasing order in infinity. To show this fact we rely on the simple observation. Indeed, let for  $L^{p(\cdot)}(\mathbb{R})$  there exists family  $l = \{l_Q\}_{Q \in \mathcal{Y}^n}$  of BSSs for which  $L^{p(\cdot)}(\mathbb{R}^n)$  satisfies uniformly lower and upper  $l$ - estimate and  $\omega : \mathbb{R} \rightarrow \mathbb{R}$ ;  $\omega(-\infty) = -\infty$ ,  $\omega(+\infty) = +\infty$  is strictly increasing absolutely continuous mapping. Then there exists family  $l_\omega$  of BSSs for which  $L^{p(\omega(\cdot))}(\mathbb{R})$  satisfies uniformly lower and upper  $l_\omega$ - estimate (see [19]).

Consider the exponent from Theorem 2.4. Let  $t_k = e^{k^3}$ ,  $m_k = e^{k^3} e^{1/k^2}$ ,  $k \geq 1$ . Let us construct new points  $t'_k, m'_k$ ,  $k \geq 1$  so that  $m'_k - t'_k = m_k - t_k$  and  $t'_{k+1} > m'_k$ . Let us now construct the pairwise linear continuous function  $\omega$  in the following way:  $\omega(x) = x$  if  $x \leq 0$ ,  $\omega(t_k) = t'_k$ ,  $\omega(m_k) = m'_k$ ;  $k \geq 1$ . We can choose the points  $t'_k, m'_k$  so that  $(m'_{k+1} - t'_k)/(m_{k+1} - t_k)$  was arbitrary large. Note that exponents  $p(w^{-1}(\cdot))$  and  $p(\cdot)$  has the same local behavior but the decreasing order in infinity of  $p(w^{-1}(\cdot))$  is very slow.

Let now consider the case  $n \geq 2$ . Let  $D = \cup_{k=1}^{\infty} [2k-1, 2k] \times [0, 1]^{n-1}$ . Consider non-trivial exponent  $p(\cdot)$  that satisfies global log-Hölder condition and is constant on the set  $\mathbb{R}^n \setminus D$ .

Let  $\{m_k\}$  be the strictly increasing sequence of integers. Consider the bijection  $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that for each  $k \in \mathbb{N}$  has the form  $\omega(x) = x - (m_k, 0, \dots, 0)$  on the set  $[2k-1, 2k] \times [0, 1]^{n-1}$ . We can choose the sequence  $\{m_k\}$  so that  $p(\omega(\cdot)) \in \mathcal{B}(\mathbb{R}^n)$  but  $p(\omega(\cdot)) \notin \mathcal{B}^{loc}(\mathbb{R}^n)$ .  $\square$

Note that only the condition  $dx \in A_{p(\cdot)}^{loc}$  (even  $dx \in A_{p(\cdot)}$ ) does not guarantee in general  $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$ . The corresponding example see in [18].

### 3. SOME APPLICATIONS

In this section, we give Littlewood-Paley square-function characterization of  $L^{p(\cdot)}(\mathbb{R}^n)$  when  $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$ . Let us recall the definition of local Muckenhoupt

weights. The weight class  $A_p^{loc}$  ( $1 < p < \infty$ ) consists of all nonnegative locally integrable functions  $w$  on  $\mathbb{R}^n$  for which

$$A_p^{loc}(w) = \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q w(x) dx \left( w(x)^{-p'/p} dx \right)^{p/p'} < \infty.$$

Extending the suprema from  $|Q| \leq 1$  to all  $Q$  gives the definition of the usual classes  $A_p$ . It follows directly from definition that  $A_p \subset A_p^{loc}$ . The littlewood-Paley theory for weight lebesgue space  $L_w^p$  with local Muckenhoupt weights was investigated by Rychkov in [26]. For more details for  $A_p^{loc}$  weights we refer paper [26].

Below we formulate analog of Rubio de Francia theorem for variable exponent case. Hereafter,  $\mathcal{F}$  will denote a family of ordered pairs of non-negative, measurable functions  $(f, g)$ . If we say that for some  $p$ ,  $1 < p < \infty$ , and  $w \in A_p^{loc}$

$$(3.1) \quad \int_{\mathbb{R}^n} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) dx, \quad (f, g) \in \mathcal{F},$$

we mean that this inequality holds for any  $(f, g) \in \mathcal{F}$  such that the left-hand side is finite, and that the constant  $C$  depends only on  $p$  and the constant  $A_p^{loc}(w)$ .

**Theorem 3.1.** . *Given a family  $\mathcal{F}$ , assume that (3.1) holds for some  $1 < p_0 < \infty$ , for every weight  $\omega \in A_{p_0}^{loc}$  and for all  $(f, g) \in \mathcal{F}$ . Let  $p(\cdot)$  be such that there exists  $1 < p_1 < p_-$ , with  $(p(\cdot)/p_1)' \in \mathcal{B}^{loc}(\mathbb{R}^n)$ . Then*

$$\|f\|_{p(t)} \leq C \|g\|_{p(t)}$$

for all  $(f, g) \in \mathcal{F}$  such that  $f \in L^{p(t)}(\mathbb{R}^n)$ . Furthermore, for every  $0 < q < \infty$  and sequence  $\{(f_j, g_j)\}_j \subset \mathcal{F}$ ,

$$\left\| \left( \sum_j (f_j)^q \right)^{1/q} \right\|_{p(t)} \leq C \left\| \left( \sum_j (g_j)^q \right)^{1/q} \right\|_{p(t)}.$$

In case when  $w \in A_{p_0}$  and  $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$  Theorem 3.1 proved in [4] (Theorem 1.3) (see also proof Theorem 3.25 in [6]). Note that the collection of all cubes  $Q$  with  $|Q| \leq 1$  form the Muckenhoupt basis, that is for each  $p$ ,  $1 < p < \infty$ , and for every  $w \in A_p^{loc}$ , the maximal operator  $M_{loc}$  is bounded on  $L_w^p(\mathbb{R}^n)$  ([26], lemma 2.11). The theorem 3.1 follows from theorem 2.2 and extrapolation theorem for general Banach function spaces ([6], Theorem 3.5).

We give a number of applications of Theorem 3.1. It is well known (see [26]) that for  $1 < p < \infty$  and for  $w \in A_p^{loc}$ ,

$$\int_{\mathbb{R}^n} M^{loc} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} f(x)^p w(x) dx.$$

From Theorem 3.1 with the pairs  $(M^{loc} f, |f|)$ , we get vector-valued inequalities for  $M^{loc}$  on  $L^{p(\cdot)}(\mathbb{R}^n)$ , provided there exists  $1 < p_1 < p_-$  with  $(p(\cdot)/p_1)' \in \mathcal{B}^{loc}(\mathbb{R}^n)$ ; by Theorem 2.2, this is equivalent to  $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$ . We obtain following local version of the Fefferman-Stein vector-valued maximal theorem:

**Corollary 3.2.** *Let  $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$ . Then for all  $1 < q < \infty$ ,*

$$\left\| \left( \sum_j (M^{loc} f_j)^q \right)^{1/q} \right\|_{p(\cdot)} \leq C \left\| \left( \sum_j (g_j)^q \right)^{1/q} \right\|_{p(\cdot)}.$$

Let  $1 < p < \infty$  and  $w \in A_p^{loc}$ . Let  $\varphi_0 \in C_0^\infty$  have nonzero integral, and  $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$ ,  $x \in \mathbb{R}^n$ . Consider the square operator  $S = S_{\varphi_0, \varphi}$  given by

$$(3.2) \quad S(f) = \left( \sum_{j=0}^{+\infty} |\varphi_j * f|^2 \right)^{1/2} \quad (f \in L_w^p(\mathbb{R}^n)),$$

where  $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ ,  $j \in \mathbb{N}$ . Then

$$\|S(f)\|_{L_w^p} \approx \|f\|_{L_w^p}, \quad \text{all } f \in L_w^p(\mathbb{R}^n).$$

(For details, see [26]). Therefore by theorem 3.1 we have following Littlewood-Paley square-function characterization of  $L^{p(\cdot)}(\mathbb{R}^n)$ .

**Corollary 3.3.** *Let  $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$ . Let  $\varphi_0 \in C_0^\infty$  have nonzero integral, and  $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$ . Consider the square operator  $S = S_{\varphi_0, \varphi}$  given by equation (3.2). Then*

$$\|S(f)\|_{p(\cdot)} \approx \|f\|_{p(\cdot)}, \quad \text{all } f \in L^{p(\cdot)}(\mathbb{R}^n).$$

#### 4. APPENDIX

Let  $\varphi(x, t) = t^{p(x)}$   $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $1 < p_- \leq p_+ < \infty$ . We need some notations. For  $t \geq 0$ ,  $s \geq 1$ , we define

$$\varphi(f)(x) : \mathbb{R}^n \rightarrow [0, +\infty) = \mathbb{R}^{\geq 0}, \quad (\varphi(f))(x) = \varphi(x, |f(x)|),$$

$$M_{s, Q\varphi} : \mathbb{R}^n \rightarrow \mathbb{R}^{\geq 0}, \quad M_{s, Q\varphi}(t) = \left( \frac{1}{|Q|} \int_Q (\varphi(x, t))^s dx \right)^{1/s}$$

$$M_{Q\varphi} : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}, \quad M_{Q\varphi}(t) = (M_{1, Q\varphi})(t).$$

Analogously we will use notation for the complementary function of  $\varphi$  given by  $\varphi^*(x, t) = (p(x) - 1)p(x)^{-p'(x)}t^{p'(x)}$ .

Note that for all cube  $Q$  functions  $(M_{s, Q\varphi})(t)$ ,  $(M_{s, Q\varphi^*})(t)$  are  $N$ -functions and satisfy uniformly  $\Delta_2$ -condition with respect to  $Q$  (see [8], Lemma 3.4). In addition we mention following properties of functions defined above ([8], Lemma 3.7): let  $s \geq 1$  and  $Q \in \mathcal{X}^n$ , then for all  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  there holds

$$(4.1) \quad (M_{s, Q\varphi^*})^* \left( \frac{1}{2} M_{s, Q\varphi} f \right) \leq M_{s, Q}(\varphi(f)).$$

Especially, for all  $u > 0$

$$(4.2) \quad (M_{s, Q\varphi^*})^* \left( \frac{1}{2} u \right) \leq M_{s, Q\varphi}(u).$$

On the other hand for all  $t > 0$  the function  $f_t = \chi_Q \varphi^*(t)/t$  satisfies

$$(4.3) \quad (M_{s, Q\varphi^*})^*(2M_{s, Q}f_t) \geq M_{s, Q}(\varphi(f_t)).$$

For  $Q \in \mathcal{Y}^n$  we define the space  $l^{|Q|} M_{Q\varphi}(Q)$



$$l^{|Q|^{M_{Q\varphi}}}(\mathcal{Q}) = \left\{ \bar{t} = \{t_Q\}_{Q \in \mathcal{Q}} : \sum_{Q \in \mathcal{Q}} |Q|(M_{Q\varphi})(t_Q) < \infty \right\},$$

equipped with the norm

$$\|\bar{t}\|_{l^{|Q|^{M_{Q\varphi}}}(\mathcal{Q})} = \inf \left\{ \lambda > 0 : \sum_{Q \in \mathcal{Q}} |Q|(M_{Q\varphi})(t_Q/\lambda) < 1 \right\}.$$

Analogously we define the spaces  $l^{|Q|^{M_{Q\varphi^*}}}(\mathcal{Q})$ ,  $l^{|Q|^{M_{s,Q\varphi}}}(\mathcal{Q})$ ,  $l^{|Q|^{M_{s,Q\varphi^*}}}(\mathcal{Q})$ .

**Definition 4.1.** Let

$$l^{|Q|^{(M_{Q\varphi^*})^*}}(\mathcal{Q}) \hookrightarrow l^{|Q|^{M_{Q\varphi}}}(\mathcal{Q})$$

are uniformly continuous with respect to  $\mathcal{Q} \in \mathcal{Y}_{loc}^n$  ( $\mathcal{Q} \in \mathcal{Y}^n$ ) i.e. for all  $A_1 > 0$  there exists  $A_2 > 0$  such that for all  $\mathcal{Q} \in \mathcal{Y}_{loc}^n$  (all  $\mathcal{Q} \in \mathcal{Y}^n$ ) and all sequences  $\{t_Q\}_{Q \in \mathcal{Q}}$  there holds

$$\sum_{Q \in \mathcal{Q}} |Q|(M_{Q\varphi^*})^*(t_Q) \leq A_1 \Rightarrow \sum_{Q \in \mathcal{Q}} |Q|(M_{Q\varphi})(t_Q) \leq A_2.$$

Then we say that  $M_{Q\varphi}$  is locally dominated (dominated) by  $(M_{Q\varphi^*})^*$  and write  $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*(loc)$  ( $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*$ ).

Analogously we may define uniformly continuous embedding discrete function spaces defined above with respect to  $\mathcal{Q} \in \mathcal{Y}_{loc}^n$  ( $\mathcal{Q} \in \mathcal{Y}^n$ ). The basic property of domination ( $\preceq$ ) in a "pointwise" sense is described in original paper [8]. analogous properties of local domination is essentially based on the following general lemma (note that if  $X = \mathcal{X}^n(loc)$  and  $Y = \mathcal{Y}^n(loc)$ , then  $X, Y$  are admissible for Lemma 4.2 )

**Lemma 4.2.** ([8], Lemma 7.1) *Let  $X$  be an arbitrary set. Let  $Y$  be a subset of the power set of  $X$  such that  $M_1 \subset M_2 \in Y$  implies  $M_1 \in Y$ . Let  $\psi_1, \psi_2 : X \rightarrow \mathbb{R}^{\geq}$ . If there exists  $A_1 > 0$  and  $A_2, A_3 \geq 0$  such that for all  $M \in Y$*

$$\omega\psi_1(\omega) \leq A_1 \Rightarrow \sum_{\omega \in M} \psi_2(\omega) \leq A_2 \sum_{\omega \in M} \psi_1(\omega) + A_3$$

*then there exists  $b : X \rightarrow \mathbb{R}^{\geq}$  such that for all  $\omega \in X$  holds*

$$(4.4) \quad \psi_1(\omega) \leq \frac{A_1}{4} \Rightarrow \psi_2 \leq \max \left\{ \frac{4A_3}{A_1}, 2A_2 \right\} \psi_1 + b(\omega)$$

*and*

$$(4.5) \quad \sup_{M \in Y} \sum_{\omega \in M} b(\omega) \leq A_3.$$

*If on the other hand there exist  $b : X \rightarrow \mathbb{R}^{\geq}$ ,  $A_1 > 0$ , and  $A_2, A_3 \geq 0$  such that (4.4) and (4.5) hold, then for all  $M \in Y$*

$$\psi_1(\omega) \leq \frac{A_1}{4} \Rightarrow \sum_{\omega \in M} \psi_2(\omega) \leq \max \left\{ \frac{4A_3}{A_1}, 2A_2 \right\} \omega\psi_1(\omega) + A_3.$$

We can now state characterization of classes  $\mathcal{A}^{loc}$  and  $\mathcal{A}$ .

**Theorem 4.3.** *Exponent  $p(\cdot)$  is of class  $\mathcal{A}^{loc}$  (of class  $\mathcal{A}$ ) if and only if  $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*(loc)$  ( $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*$ )*

The proof of above theorem in case  $p(\cdot)$  is of class  $\mathcal{A}$  is based on properties (4.1)-(4.3) of  $M_{Q\varphi}$  and  $(M_{Q\varphi^*})$  and may use analogously arguments in local variant.

Inspired by the classical Muckenhoupt class  $A_\infty$  in [8] was defined condition  $\mathcal{A}_\infty$ . The importance of our considerations is analogous of definition in local case.

**Definition 4.4.** We say that exponent  $p(\cdot)$  is of class  $\mathcal{A}_\infty^{loc}$  (class  $\mathcal{A}_\infty$ ) if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that the following holds: if  $N \subset \mathbb{R}^n$  is measurable and  $\mathcal{Q} \in \mathcal{Y}_{loc}^n$  ( $\mathcal{Q} \in \mathcal{Y}^n$ ) such that

$$|Q \cap N| \geq \varepsilon |Q| \quad \text{for all } Q \in \mathcal{Q},$$

then for any sequence  $\{t_Q\}_{Q \in \mathcal{Q}}$

$$\delta \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right\|_{p(\cdot)} \leq \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_{Q \cap N} \right\|_{p(\cdot)}.$$

It is not hard to proof that if exponent  $p(\cdot)$  is of class  $\mathcal{A}^{loc}$  then exponent  $p(\cdot)$  is in  $\mathcal{A}_\infty^{loc}$ .

The important property of exponents from class  $\mathcal{A}_\infty$  is that  $\mathcal{A}_\infty$  implies  $M_{s,Q\varphi} \preceq M_{Q\varphi}$  for some  $s > 1$ . The proof of this result is based on the following lemma.

**Lemma 4.5.** ([8] Lemma 5.5) *Let exponent  $p(\cdot)$  is of class  $\mathcal{A}_\infty$ . Then there exists  $\delta > 0$  and  $A \geq 1$  such that for all  $Q \in \mathcal{Y}^n$ , all  $\{t_Q\}_{Q \in \mathcal{Q}}, t_Q \geq 0$ , and all  $f \in L_{loc}^1$  with  $M_Q f \neq 0$ ,  $Q \in \mathcal{Q}$ , holds*

$$\left\| \sum_{Q \in \mathcal{Q}} t_Q \left| \frac{f}{M_Q f} \right|^\delta \chi_Q \right\|_{p(\cdot)} \leq A \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right\|_{p(\cdot)}.$$

Note that a very similar argument can be used to obtain local version of Lemma 4.5. In original proof of Lemma 4.5 is used  $Q$ -dyadic ( $Q \in \mathcal{X}^n$ ) maximal function  $M^{\Delta,Q}$  ([8], Definition 5.4). Note that in fact in proof of Lemma 4.5 it is used local  $Q$ -dyadic maximal function, where the supremum is taken over all  $Q$ -dyadic cube  $Q'$  containing  $x$  and  $|Q'| \leq |Q|$ . As a consequence of local variant of Lemma 4.5 we obtain a kind reverse Hölder estimate for exponents from class  $\mathcal{A}^{loc}$ .

**Theorem 4.6.** *Let  $p(\cdot) \in \mathcal{A}^{loc}$ . Then there exists  $s > 1$ , such that  $M_{s,Q\varphi} \preceq M_{Q\varphi}(loc)$ .*

From Theorem 4.3 and Theorem 4.6 for local variant we obtain

**Theorem 4.7.** *The following conditions are equivalent*

- (a)  $p(\cdot)$  is of class  $\mathcal{A}^{loc}$ .
- (b)  $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*(loc)$
- (c) *There exists  $s > 1$ , such that  $M_{s,Q\varphi} \preceq M_{Q\varphi} \preceq (M_{Q\varphi^*})^* \preceq (M_{s,Q\varphi^*})^*(loc)$ .*

The key lemma from which was derived original Theorem 3.1 is Lemma 8.7 from [8]. We formulate analogous statement for local variant.

**Lemma 4.8.** *Let  $p(\cdot) \in \mathcal{A}^{loc}$ . Then there exists  $s > 1$  such that for all  $A_1 > 0$  there exist  $A_2 > 0$  such that the following holds:*

*For all families  $\mathcal{Q}_\lambda \in \mathcal{Y}_{loc}^n$ ,  $\lambda > 0$ , with*

$$\sum_{Q \in \mathcal{Q}_\lambda} |Q| (M_{s,Q\varphi^*})^*(\lambda) \leq A_1$$

and

$$\int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_\lambda} |Q|(M_{s,Q\varphi^*})^*(\lambda) \leq A_1,$$

there holds

$$\int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_\lambda} |Q|(M_{s,Q\varphi})(\lambda) \leq A_2.$$

Note that relation described in Lemma 8.7 from [8] is denoted as  $M_{Q\varphi} \ll (M_{s,Q\varphi^*})^*$  (strong domination).

The proof of Lemma 4.8 is based on some pointwise estimate of functions  $(M_{Q\varphi^*})^*$  and  $(M_{s,Q\varphi^*})^*$ . This properties we will describe bellow in Lemma 4.9, 4.10.

If  $p(\cdot) \in \mathcal{A}^{loc}$ , then  $M_{s,Q\varphi} \preceq (M_{s,Q\varphi^*})^*(loc)$  for some  $s > 1$ . It is not hard to prove that (analogously as the proof of Lemma 8.3 from [8]) uniformly in  $Q \in \mathcal{X}_{loc}^n$

$$(4.6) \quad |Q|(M_{s,Q\varphi})\left(\frac{1}{\|\chi_Q\|_{p(\cdot)}}\right) \sim 1, \quad |Q|(M_{s,Q\varphi^*})^*\left(\frac{1}{\|\chi_Q\|_{p(\cdot)}}\right) \sim 1.$$

It is important to investigate for any  $Q \in \mathcal{X}_{loc}^n$  the function

$$\alpha_s(Q, t) = \frac{(M_{s,Q\varphi})(t)}{(M_{s,Q\varphi^*})^*(t)}.$$

**Lemma 4.9.** *Let  $p(\cdot) \in \mathcal{A}^{loc}$ . Then uniformly in  $Q \in \mathcal{X}_{loc}^n$  and  $t > 0$*

$$\alpha_s(Q, 1/\|\chi_Q\|_{p(\cdot)}) \sim 1, \quad \alpha_s(Q, 1) \sim 1.$$

Moreover, there exists  $C \geq 1$  such that for all  $Q \in \mathcal{X}_{loc}^n$

$$\alpha_s(Q, t_2) \leq C(\alpha_s(Q, t_1) + 1) \quad \text{for } 0 < t_1 \leq t_2 \leq 1,$$

$$\alpha_s(Q, t_3) \leq C(\alpha_s(Q, t_4) + 1) \quad \text{for } 1 < t_3 \leq t_4 \leq 1.$$

Furthermore, for all  $C_1, C_2 > 0$  there exists  $C_3 \geq 1$  such that for all  $Q \in \mathcal{X}_{loc}^n$

$$(4.7) \quad t \in \left[ C_1 \min \left\{ 1, \frac{1}{\|\chi_Q\|_{p(\cdot)}} \right\}, C_1 \max \left\{ 1, \frac{1}{\|\chi_Q\|_{p(\cdot)}} \right\} \right] \Rightarrow \alpha_s(Q, t) \leq C_3.$$

The proof of analogous statement for nonlocal case ([8], Lemma 8.4) is based on the estimates (4.6) and some properties (not depend on  $Q$ ) of convex functions  $M_{s,Q\varphi}, (M_{s,Q\varphi^*})^*$ . This arguments may use in local variant.

**Lemma 4.10.** *Let  $p(\cdot) \in \mathcal{A}(loc)$ . Then there exists  $b : \mathcal{X}^n(loc) \rightarrow \mathbb{R}^\geq$  and  $K > 0$  such that*

$$\sup_{Q \in \mathcal{Y}^n(loc)} \sum_{Q \in \mathcal{Q}} |Q|b(Q) + \sup_{Q \in \mathcal{X}^n(loc)} |Q|b(Q) < \infty$$

and for all  $Q \in \mathcal{X}^n(loc)$  and all  $t \geq 0$  holds

$$|Q|(M_{s,Q\varphi^*})^*(t) \leq 1 \Rightarrow (M_{s,Q\varphi})(t) \leq K(M_{s,Q\varphi^*})^*(t) + b(Q).$$

Moreover, for all  $Q \in \mathcal{X}^n(loc)$  and all  $t \geq 1$  there holds

$$|Q|(M_{s,Q\varphi^*})^*(t) \leq 1 \Rightarrow (M_{s,Q\varphi})(t) \leq K(M_{s,Q\varphi^*})^*(t).$$

The proof may be obtained from general Lemma 4.1 and by using (4.7) estimate (see [8], proof Lemma 8.5).

**Lemma 4.11.** *Assume  $M_{s_2, Q\varphi} \preceq M_{s_2, Q\varphi^*}^*(loc)$  for some  $s_2 > 1$  and  $1 \leq s_1 \leq s_2$ . Then uniformly in  $Q \in \mathcal{X}^n(loc)$  and  $t > 0$*

$$\left( \alpha_{s_2}(Q, t^{\frac{s_1}{s_2}}) \right)^{\frac{s_2}{s_1}} \sim \alpha_{s_1}(Q, t).$$

The proof of Lemma 4.11 basically based on the Lemma 4.10 and may proof as analogous lemma from [8] (Lemma 8.6).

Let  $f$  be locally integrable function. For  $q \geq 1$  we consider the local maximal operator given by

$$M_q^{loc} f(x) = \sup_{Q \ni x, |Q| \leq 1} \left( \frac{1}{|Q|} \int_Q |f(y)|^q dy \right)^{1/q}.$$

We define local dyadic maximal operator  $M_{q,d}^{loc}$  witch restricted supremum in definition of  $M_q^{loc}$  by dyadic cubes (cubes of the form  $Q = 2^{-z}((0,1)^n + k)$ ,  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ ,  $z \in \mathbb{N}_0$ ).

For fixed  $t \in \mathbb{R}^n$  we define also maximal operator  $M_{q,d}^{loc,t}$  which restricted supremum in definition of  $M_{q,d}^{loc}$  on the cubes  $Q - t$ , where  $Q$  dyadic cubes.

Note that there is a constant  $C > 0$  such that (see [27])

$$(4.8) \quad M_q^{loc} f(x) \leq C \int_{[-4,4]^n} M_{q,d}^{loc,t} f(x) dt.$$

The main step to proof Theorem 2.2 (as in proof of original Theorem 2.1) is following Theorem.

**Theorem 4.12.** *Let  $p(\cdot) \in \mathcal{A}_{loc}$ . Then there exists  $q > 1$  such that  $M_q^{loc}$  is continuous on  $L^{p(\cdot)}(\mathbb{R}^n)$ .*

Note that using (4.8) estimate it is sufficiently to proof Theorem 4.12 for operator  $M_{q,d}^{loc}$ .

It is suffices to show that there exists  $A > 0$  such that for all  $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq 1 \Rightarrow \int_{\mathbb{R}^n} |M_{q,d}^{loc} f(x)|^{p(x)} dx \leq A.$$

For  $\lambda > 0$  define functions

$$f_{0,\lambda} = f \chi_{\{|f| \leq \lambda\}}, \quad f_{1,\lambda} = f \chi_{\{|f| > \lambda\}}.$$

Then

$$\{M_{q,d}^{loc} f > \lambda\} \subset \{M_{q,d}^{loc} f_{0,\lambda} > \lambda/2\} \cup \{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\}.$$

This implies

$$\begin{aligned} \int_{\mathbb{R}^n} |M_{q,d}^{loc} f(x)|^{p(x)} dx &= \int_0^\infty \int_{\mathbb{R}^n} p(x) \lambda^{p(x)-1} \chi_{\{M_{q,d}^{loc} f > \lambda\}} dx d\lambda \\ &\leq C \sum_{j=1}^2 \int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{j,\lambda} > \lambda/2\}} dx d\lambda. \end{aligned}$$

For  $\lambda > 0$  let  $\mathcal{Q}_{0,\lambda}$  be the decomposition of  $\{M_{q,d}^{loc} f_{0,\lambda} > \lambda/2\}$  into maximal dyadic cubes. Then for all  $Q \in \mathcal{Q}_{0,\lambda}$  there holds (uniformly in  $Q$ )

$$M_{q,Q} f_{0,\lambda} \sim \lambda$$

and we have

$$\int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{0,\lambda} > \lambda/2\}} dx d\lambda \leq C \int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{0,\lambda}} |Q|(M_{Q\varphi})(\lambda) d\lambda.$$

Denote  $f_{1,\lambda}^k = \chi_{(0,1)^n+k} f_{1,\lambda}$ ,  $k \in \mathbb{Z}^n$ . Note that if  $x \in (0,1)^n + k$  then

$$M_{q,d}^{loc} f_{1,\lambda}(x) = M_{q,d}^{loc} f_{1,\lambda}^k(x)$$

and

$$\{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\} = \cup_{k \in \mathbb{Z}^n} \{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}.$$

We have

$$\begin{aligned} & \int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\}} dx d\lambda \\ &= \int_0^\infty \lambda^{-1} \sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n+k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda. \end{aligned}$$

Define  $m_k = 2 \int_{(0,1)^n+k} |f(x)| dx$ , we have

$$\begin{aligned} & \int_0^\infty \lambda^{-1} \int_{(0,1)^n+k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda \\ &= \int_0^{m_k} \lambda^{-1} \int_{(0,1)^n+k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda \\ &+ \int_{m_k}^\infty \lambda^{-1} \int_{(0,1)^n+k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda. \end{aligned}$$

Note that

$$\int_0^{m_k} \lambda^{-1} \int_{(0,1)^n+k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda \leq C \int_{(0,1)^n+k} \left( \int_{(0,1)^n+k} |f(t)| dt \right)^{p(x)} dx.$$

Let  $\mathcal{Q}_{1,\lambda}^k$  be the decomposition of  $\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}$  into maximal dyadic cubes. Then for all  $Q \in \mathcal{Q}_{1,\lambda}^k$  there holds

$$M_{q,Q} f_{1,\lambda}^k = M_{q,Q} f_{1,\lambda} \sim \lambda.$$

Define  $\mathcal{Q}_{1,\lambda} = \cup_{k \in \mathbb{Z}^n} \mathcal{Q}_{1,\lambda}^k$ . Then we have

$$\begin{aligned} & \int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\}} dx d\lambda \\ & \leq C \sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n+k} \left( \int_{(0,1)^n+k} |f(t)| dt \right)^{p(x)} dx + \int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{1,\lambda}} |Q|(M_{Q\varphi})(\lambda) d\lambda. \end{aligned}$$

For first term we have

$$\sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n+k} \left( \int_{(0,1)^n+k} |f(t)| dt \right)^{p(x)} dx \leq C.$$

The second term  $\int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{1,\lambda}} |Q|(M_{Q\varphi})(\lambda) d\lambda$  can be estimated in the same way as in the Theorem 6.2 from [8].

## REFERENCES

- [1] E. Acrebi and G. Mingione, Regularity results for a class of functionals with nonstandard growth. *Arch. Ration. Mech. Anal.* 1569 (2001), 121–140.
- [2] E. Acrebi and G. Mingione, Regularity results for stationary electro-rheological fluids: the stationary case, *C.R. Acad. Sci. Paris, Ser. I* 334(2002), no.9, 817–822.
- [3] Y. Chen, S. Levine and R. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* 66 (2006), no. 4, 1383–1406.
- [4] D. Cruz-Uribe, A. Fiorenza, J.M.Martell and C. Pérez, The boundedness of classical operators on variable  $L^p$  spaces, *Ann. Acad. sci. Fenn.Math.* 31(2006), 239–264.
- [5] D. Cruz-Uribe, A. Fiorenza and C. Neugebauer, The maximal function on variable  $L^p$  spaces. *Ann. Acad. Sci. Fen. Math. J.* 28 (2003), 223–238, and 29 (2004), 247–249.
- [6] D. Cruz-Uribe, J.M.Martell and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, Birkhäuser, 2011.
- [7] L. Diening, Maximal function on generalized Lebesgue spaces, *Math. Inequal. Appl.* 7 (2004), no. 2, 245–253.
- [8] L. Diening, Maximal function on Orlicz-Musielak spaces and genralizd Lebesgue spaces. *Bull. Sc. Math.* 129 (2005), pp. 657–700.
- [9] L. Diening, P. Hästö, P. Harjulehto and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer Lecture Notes, vol. 2017, Springer-Verlag, Berlin 2011.
- [10] X.-L. Fan, Global  $C^{1;\alpha}$  regularity for variable exponent elliptic equations in divergence form, *J. Differential Equations* 235 (2007), no. 2, 397–417.
- [11] R. Fortini, D. Mugnai and P. Pucci, Maximum principles for anisotropic elliptic inequalities, *Non-linear Anal.* 70 (2009), no. 8, 2917–2929.
- [12] P. A. Hästö, Local-to-global results in variable exponent spaces. *Math. Res. Lett.* 16 (2009), 263–278.
- [13] P. Harjulehto, P. Hästö, V. Latvala, O. Toivanen, Critical variable exponent functionals in image restoration, *Appl. Math. Letters* 26 (2013), 56–60.
- [14] E. Kapanadze and T. Kopaliani, A note on maximal operator on  $L^{p(\cdot)}(\Omega)$  spaces. *Georgian Math. J.* 16 (2008), no. 2, 307–316.
- [15] T. Kopaliani, On some structural properties of Banach function spaces and boundedness of certain integral operators. *Czechoslovak Math. J.* 54 (2004), 791–805.
- [16] T. Kopaliani, Infimal convolution and Muckenhoupt  $A_{p(\cdot)}$  condition in variable  $L^p$  spaces. *Arch. Math. (Basel)* 89 (2007), 185–192.
- [17] T. Kopaliani, Greediness of the wavelet system in variable Lebesgue spaces. *East J. Approx.* 14 (2008), 29–37.
- [18] T. Kopaliani, On the Muckenchaup condition in variable Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* 148 (2008), 29–33.
- [19] T. Kopaliani, A characterization of some weighted norm inequalities for maximal operators. *Z. Anal. Anwend.* 29 (2010), 401–412.
- [20] A. Lerner, On some questions related to the maximal operator on variable  $L^p$  spaces. *Trans. Amer. Math. Soc.* 362 (2010), 4229–4242.
- [21] F. Li, Z. Li and L. Pi, Variable exponent functionals in image restoration, *Appl. Math. Comput.* 216 (2010), no. 3, 870–882.
- [22] A. Nekvinda, Equivalence of lpn norms and shift operators. *Math. Inequal. Appl.* 5 (2002), 711–723.
- [23] A. Nekvinda, Hardy-Littlewood maximal operator on  $L^{p(x)}(R^n)$ . *Math. Inequal. Appl.* 7 (2004), 255–266.
- [24] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, 1748, Springer-Verlang, Berlin, 2000.
- [25] M. Růžička, Modeling, mathematical and numerical analysis of electrorheological fluids, *appl. Math.* 49(2004), no. 6, 565–609.
- [26] V. S. Rychkov, Littlewood-Paley Theory and function spaces with  $A_p^{loc}$  weights. *Math. Nach.* 224(2001), 145–180.
- [27] E. T. Sawyer, Acharacterization for two weight norm inequalities for maximal operators, *Studia Math.* 75(1982), 1–11.

INSTITUTE OF MATHEMATICS OF THE ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, ŽITNA  
25, 11567 PRAHA 1, CZECH REPUBLIC  
*E-mail address:* `gogatish@math.cas.cz`

FACULTY OF EXACT AND NATURAL SCIENCES, TBILISI STATE UNIVERSITY, CHAVCHAVADZE ST.1,  
TBILISI 0128 GEORGIA  
*E-mail address:* `tengiz.kopaliani@tsu.ge`

FACULTY OF EXACT AND NATURAL SCIENCES, TBILISI STATE UNIVERSITY, CHAVCHAVADZE ST.1,  
TBILISI 0128 GEORGIA  
*E-mail address:* `ana.danelia@tsu.ge`